

# On the transfer matrix of a MIMO system

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## Abstract

We develop a deterministic ab-initio model for the input-output relationship of a multiple-input multiple-output (MIMO) wireless channel, starting from the Maxwell equations combined with Ohm's Law. The main technical tools are scattering and geometric perturbation theories. The derived relationship can lead us to a deep understanding of how the propagation conditions and the coupling effects between the elements of multiple-element arrays affect the properties of a MIMO channel, e.g. its capacity and its number of degrees of freedom.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Main technical assumptions . . . . .	3
<b>2</b>	<b>The Maxwell equations in the frequency domain</b>	<b>4</b>
2.1	The magnetic vector potential method . . . . .	4
2.2	The transfer matrix . . . . .	5
<b>3</b>	<b>Decoupling the receivers and transmitters from the environment</b>	<b>6</b>
3.1	The study of $\chi_{R_a}(H - z)^{-1}\chi_{T_a}$ . . . . .	9
3.2	The main theorem . . . . .	10
3.3	Application: the spread function in the case of distant scatterers . . . . .	11
<b>4</b>	<b>Conclusions and open problems</b>	<b>14</b>
<b>5</b>	<b>Appendix: A Lippmann-Schwinger type equation for the resolvent</b>	<b>15</b>
5.1	The case of just one scatterer . . . . .	15
5.2	The case of several scatterers . . . . .	16
<b>6</b>	<b>Acknowledgments</b>	<b>17</b>

## 1 Introduction

In wireless communication, systems with multiple antenna arrays at both ends of the link are called MIMO (multiple input multiple output). The main interest is to compute the number of bits/s which can be transmitted by these systems. It has been shown that this quantity, called channel capacity [1, 2, 3, 4, 5], is linked with the transfer matrix  $\mathcal{H}$  whose element  $\mathcal{H}_{mn}$  is the ratio between the complex current intensity  $I_m$  at the load of the  $m^{th}$  receiving (RX) antenna over the current  $I_n$  which feeds the  $n^{th}$  transmitting (TX) antenna. If  $M$  and  $N$  are respectively the number of RX antennas and TX antennas, then  $\mathcal{H}$  is a  $M \times N$  matrix.

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Even if one can experimentally determine  $\mathcal{H}$  in a given environment when the antennas are fixed, its entries show such large variations when moving the antennas arrays from place to place that the knowledge of  $\mathcal{H}$  for some number of locations seems to be insufficient for having an idea of the capacity everywhere. Fortunately, it seems that the eigenvalue distribution of the matrix  $\mathcal{H}^*\mathcal{H}$  (here  $*$  means adjoint) is more robust with respect to the displacements, but this has to be better understood. In order to mimic the variations of the entries of  $\mathcal{H}$ , one possibility is to replace  $\mathcal{H}$  by random matrices introducing several distributions for their elements (see [6, 7]). In sharp contrast with this method, we try to understand the variations through the physical study of the wave propagation in a given environment. Thus we employ a deterministic modeling method and our aim is to describe  $\mathcal{H}$  by quantities linked with the physical characteristics of the antennas and the environment. But certain stochastic aspects can naturally appear if some parameters of the scatterers present in the medium are described by stochastic variables (like position, density etc). This work was motivated by the existence of an heuristic formula intensively used by the MIMO signal experts. It appeared in the works of B.Fleury et al. see e.g. [8, 9, 10, 11]; see also [12]. This formula describes the transfer matrix in the following way:

$$\mathcal{H} = \int_{S^2 \times S^2} c(\Omega_R)^t \mathcal{A}(\Omega_R, \Omega_T) c(\Omega_T) d\Omega_R d\Omega_T. \quad (1.1)$$

where  $\Omega_T = (\phi_T, \theta_T)$  is a direction of departure from the  $TX$  side where  $\phi_T, \theta_T$  are respectively its azimuth and elevation angles and  $\Omega_R = (\phi_R, \theta_R)$  is a direction of arrival at the  $RX$  side,  $c(\Omega_T)$  and  $c(\Omega_R)$  are respectively  $2 \times N$  and  $2 \times M$  matrices whose elements depend only on the corresponding antennas diagrams and the geometry of the arrays, while  $\mathcal{A}(\Omega_R, \Omega_T)$  is a  $2 \times 2$  matrix which depends only on the environment.

Our aim is to understand this formula from first principles, that is solving the Maxwell equations in a typical complex environment formed by the  $TX$  and  $RX$  antenna arrays surrounded by buildings, persons, trees ... The antennas are pieces of metal described by their geometry and by their conductivity  $\sigma$ , while the scatterers which are made of dielectric and magnetic materials can also be described by their geometry, permittivity, permeability and conductivity.

To determine  $\mathcal{H}_{mn}$  one introduces in the second member of the Maxwell equations a harmonic current density  $\tilde{\mathbf{J}}_{in}$ , in a small volume constituted by a part of the wire which connects the  $n^{th}$  signal source to the antenna. The current intensity crossing a section  $S_T$  of the wire is  $I_n = \int_{S_T} \tilde{\mathbf{J}}_{in}(x) \cdot d\mathbf{S}_T$ . If one knows the electric fields  $\mathbf{E}$  induced by this current density, in particular at a section  $S_R$  of the wire connecting the  $m^{th}$   $RX$  antenna to its load, there one can calculate the current density using the Ohm law  $\tilde{\mathbf{J}}_{out} = \sigma \mathbf{E}$  and deduce the current crossing this section,  $I_m = \sigma \int_{S_R} \mathbf{E}(x) \cdot d\mathbf{S}_R$ . Then  $\mathcal{H}_{mn} = \frac{I_m}{I_n}$ ; note that this formula is true when only the  $n^{th}$  emitting antenna is fed with current. For the general case see the discussion around (2.20).

To tackle these calculations, wave propagation experts try to address the difficulties in the following manner. They suppose that they know the electromagnetic fields (EM) fields radiated by the  $n^{th}$   $TX$  antenna in the free space. These fields are considered as incoming fields in a scattering problem where the scatterers are constituted by buildings, persons, trees... and they try to calculate the total fields produced adding the incoming fields and the scattered fields. This is in fact a difficult problem which is only solved analytically in some very simple and unrealistic situations. Practically it is solved using the geometrical optics approximation (ray tracing). Finally, the total EM fields calculated in this way are considered as incoming fields for the  $m^{th}$   $RX$  antenna and, once more, one is faced with a scattering problem where now the  $RX$  antennas play the role of scatterers and again one has to determine the new total fields, from which one obtains  $I_m$ .

One of the purposes of the paper is to justify this procedure. We succeed in proving that when the antennas are sufficiently small and spatially separated from the surrounding scatterers, the EM fields calculated with the procedure explained above appear as the first term in a series giving the exact fields. In Lemmas 3.1 and 3.2 we give an approximation for the total Green function which puts into evidence the decoupling between the  $TX$  antennas, the  $RX$  antennas and the surrounding scatterers.

Our proposed methodology is to use the vector potential formalism to solve the Maxwell equations.

Our main mathematical tool is geometric perturbation theory for closed but non-selfadjoint operators. These methods have been developed for apparently unrelated quantum scattering problems in [13, 14, 15, 16] and they can be used for describing the propagation of electromagnetic waves in frequency domain.

In Corollary 3.4 we derive a stronger version of the Fleury heuristic formula as a corollary of Theorem 3.3, in the case when the antenna arrays are far away from the surrounding scatterers. In particular, we link the heuristic "spread matrix function"  $\mathcal{A}(\Omega_R, \Omega_T)$  to the Green function of the environment alone.

In this paper the language is deliberately made free of too technical notions of functional analysis and operator theory in order to facilitate the understanding for non-mathematicians. Some notions concerning the Limiting Absorption Principle for the non self adjoint operators coming from the vector potential description of the *EM* fields are not yet proved, so this paper has to be completed in this respect in order to get rigorously the set of frequencies for which the Green operators exist when the "energy" tends to a real number, as operators in certain weighted  $L^2$  spaces.

On the other hand, this paper is a starting point to more applied engineering works since now we understand the link between the heuristic "spread function" in formula (1.1) and the scattering transfer operator of the environment.

## 1.1 Main technical assumptions

Throughout this paper the configuration space will be  $\mathbb{R}^3$ . The environment is known, which means that we assume the knowledge of the dielectric constant  $\epsilon$  and the conductivity  $\sigma$ . They are smooth functions of the position and frequency. There are no magnetic effects, that is  $\mu = \mu_0$  is constant.

Let us assume that we know the current density  $\tilde{\mathbf{J}}_{in}(\mathbf{x}, t)$  (i.e. in time domain) at the entrance of the transmitting antenna. It is completely determined by the transmission device.

**Assumption 1.1.** *The current density  $\tilde{\mathbf{J}}_{in}$  has the following generic properties:*

- *it is smooth in the time variable;*
- *it is compactly supported in  $\mathbf{x}$ ;*
- $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\mathbf{J}}_{in}(\mathbf{x}, t) e^{-j\omega t} dt =: \mathbf{J}(\mathbf{x}, \omega) = \overline{\mathbf{J}(\mathbf{x}, -\omega)}$  *is a smooth function, compactly supported in  $\omega \in [-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]$ . This equality is a consequence of the reality of  $\tilde{\mathbf{J}}_{in}(\mathbf{x}, t)$ .*

There is an inherent physical imprecision on the very notion of "external" source current, since this quantity is supposed to be completely determined by the transmitter. It is common to assume that the current densities on the wires feeding the transmitting antennas are constant on the disc of the corresponding cross sections. But this fact does not affect the conclusions of our paper.

**Assumption 1.2.** *Further technical assumptions on the environment:*

- $\epsilon(\mathbf{x}, \omega) = \overline{\epsilon(\mathbf{x}, -\omega)}$  and  $\sigma(\mathbf{x}, \omega) = \overline{\sigma(\mathbf{x}, -\omega)}$  *are smooth in  $\mathbf{x}$ . The case with piecewise constant  $\epsilon$  requires a rather complicated regularization procedure which will be considered elsewhere;*
- *the dielectric constant of the air is constant and equals  $\epsilon_0$ ;*
- 

$$\epsilon^{(r)}(\mathbf{x}) := \frac{\epsilon(\mathbf{x})}{\epsilon_0} = 1 + \delta\epsilon_T(\mathbf{x}) + \delta\epsilon_M(\mathbf{x}) + \delta\epsilon_R(\mathbf{x}), \quad (1.2)$$

*where all the relative  $\delta\epsilon$ 's are compactly supported perturbations, with disjoint supports. Here  $T$  means the transmitting antenna(s),  $R$  the receiver(s), and  $M$  the scatterer(s).*

- *the conductivity can be written*

$$\sigma(\mathbf{x}) = \sigma_T(\mathbf{x}) + \sigma_R(\mathbf{x}), \quad (1.3)$$

*where the  $\sigma$ 's are compactly supported on the regions containing the antennas.*

## 2 The Maxwell equations in the frequency domain

In the frequency domain, the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{H}$  satisfy the following system of equations:

$$\nabla \cdot \mathbf{H} = 0; \quad (2.1)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + (j\epsilon\omega + \sigma)\mathbf{E}; \quad (2.2)$$

$$\nabla \times \mathbf{E} = -j\mu_0\omega\mathbf{H}; \quad (2.3)$$

$$\rho = \nabla \cdot (\epsilon\mathbf{E}); \quad (2.4)$$

$$-j\omega\rho = \nabla \cdot (\mathbf{J} + \sigma\mathbf{E}). \quad (2.5)$$

Here we incorporate Ohm's Law in (2.2), in the sense that an electric field  $\mathbf{E}$  generates a charge-current density  $\sigma\mathbf{E}$  in metals. We stress that we do not work with perfect conducting antennas; taking  $\sigma$  to infinity is a singular operation within our formalism, and the study of this limit is a very interesting mathematical problem which is left as an open problem. We note that one can start with infinite conductivities right from the beginning, but the price is that one needs to deal with severely ill-posed inverse problems (see e.g. [17, 18]).

### 2.1 The magnetic vector potential method

Equations (2.4) and (2.5) allow us to eliminate the unknown charge density  $\rho$ :

$$\nabla \cdot \{\mathbf{J} + (j\epsilon\omega + \sigma)\mathbf{E}\} = 0. \quad (2.6)$$

Equation (2.1) allows us to represent  $\mathbf{H} = \nabla \times \mathbf{A}$ , where  $\mathbf{A}$  is a magnetic vector potential. Using this in (2.3) gives  $\nabla \times (\mathbf{E} + j\mu_0\omega\mathbf{A}) = 0$ , thus there exists a scalar potential  $\phi$  such that:

$$\mathbf{E} + j\mu_0\omega\mathbf{A} = -\nabla\phi. \quad (2.7)$$

Introducing the magnetic vector potential in (2.2) and using (2.7) we obtain:

$$\nabla(\nabla \cdot \mathbf{A}) - \Delta\mathbf{A} = \mathbf{J} + (j\epsilon\omega + \sigma)(-j\mu_0\omega\mathbf{A} - \nabla\phi). \quad (2.8)$$

Note the important thing that if we have a pair  $(\mathbf{A}, \phi)$  which solves (2.8), then (2.6) is automatically satisfied because the left hand side of (2.8) is divergence free. Thus we have the freedom of choosing a  $\phi$  completely determined by  $\mathbf{A}$  through a Lorenz-type gauge condition, and then solve (2.8) for  $\mathbf{A}$ . Denoting by  $\mathbf{A}_L$  such a special solution, we choose:

$$\phi_L := -\frac{1}{j\epsilon\omega + \sigma}(\nabla \cdot \mathbf{A}_L). \quad (2.9)$$

With this choice, (2.8) simplifies to:

$$-\Delta\mathbf{A}_L = \mathbf{J} - j\mu_0\omega(j\epsilon\omega + \sigma)\mathbf{A}_L - (\nabla \cdot \mathbf{A}_L)\frac{\nabla(j\epsilon\omega + \sigma)}{(j\epsilon\omega + \sigma)}. \quad (2.10)$$

Introducing the notation

$$k^2(\mathbf{x}) := \mu_0\epsilon(\mathbf{x})\omega^2 - j\mu_0\omega\sigma(\mathbf{x}), \quad (2.11)$$

we can rewrite (2.10) as:

$$-\Delta\mathbf{A}_L - k^2\mathbf{A}_L + (\nabla \cdot \mathbf{A}_L)\nabla \ln(k^2) = \mathbf{J}. \quad (2.12)$$

Now if we have a solution to (2.12), then the fields  $\mathbf{E}$  and  $\mathbf{H}$  are given by:

$$\begin{aligned} \mathbf{E} &= -j\mu_0\omega\mathbf{A}_L + \nabla \left( \frac{\nabla \cdot \mathbf{A}_L}{j\epsilon\omega + \sigma} \right), \\ \mathbf{H} &= \nabla \times \mathbf{A}_L. \end{aligned} \quad (2.13)$$

Denoting by  $k_0^2 := \mu_0 \epsilon_0 \omega^2$  and using  $\epsilon^{(r)}(\mathbf{x}) = \epsilon(\mathbf{x})/\epsilon_0$  (see (1.2)) we have:

$$k^2(\mathbf{x}) = k_0^2 + (\epsilon^{(r)}(\mathbf{x}) - 1)k_0^2 - j\mu_0\omega\sigma(\mathbf{x}) =: k_0^2 + \delta k^2(\mathbf{x}), \quad (2.14)$$

where  $\delta k^2(\mathbf{x}) = (\epsilon^{(r)}(\mathbf{x}) - 1)k_0^2 - j\mu_0\omega\sigma(\mathbf{x})$  is smooth and compactly supported. Note that  $k^2$  cannot be zero if  $\epsilon^{(r)} \neq 0$ .

Define the following first order differential operator  $W$  in the following way:

$$W\mathbf{A} := -\delta k^2\mathbf{A} + \{\nabla \cdot \mathbf{A}\}\nabla \ln(k^2) \quad (2.15)$$

and observe that it is relatively compact with respect to  $-\Delta$  on the Hilbert space  $[L^2(\mathbb{R}^3)]^3$ . Moreover, the operator sum  $-\Delta + W$  is closed on the Sobolev space  $[H^2(\mathbb{R}^3)]^3$ , and if  $z \in \mathbb{C}$  then we can define the inverse  $(-\Delta + W - z)^{-1}$  for at least when  $|Im(z)|$  is large enough.

## 2.2 The transfer matrix

The solution we are looking for in (2.12) is:

$$\mathbf{A}_L = (-\Delta + W - k_0^2 - j0_+)^{-1}\mathbf{J}_{\text{in}}, \quad (2.16)$$

where we have to give a rigorous sense to the limiting absorption principle. The problem is far from being trivial because the perturbation  $W$  is not symmetric; technical details and full proofs will be given elsewhere. We only want to mention that under rather general conditions, the limit in (2.12) is expected to hold outside a discrete set of frequencies. In order to simplify the notation, we stop writing  $0_+$ .

Having determined  $\mathbf{A}_L$ , we can derive  $\mathbf{E}$  from (2.13):

$$\mathbf{E} = -j\mu_0\omega(-\Delta + W - k_0^2)^{-1}\mathbf{J}_{\text{in}} + \nabla \left( \frac{1}{j\epsilon\omega + \sigma} \nabla \cdot (-\Delta + W - k_0^2)^{-1}\mathbf{J}_{\text{in}} \right). \quad (2.17)$$

Let us be more precise and assume that we have  $N \geq 1$  transmitting antennas for which we can write:

$$\mathbf{J}_{\text{in}} = \sum_{n=1}^N I_{\text{in}}^{(n)} \mathbf{J}_{\text{in}}^{(n)}, \quad (2.18)$$

where  $I_{\text{in}}^{(n)}(\omega)$  is the current running through the tranverse section of the  $n$ -th antenna, and  $\mathbf{J}_{\text{in}}^{(n)}$  is its corresponding normalized current density (i.e. the flux integral of any  $\mathbf{J}_{\text{in}}^{(n)}$  through the transverse section of antenna  $n$  equals one).

According to (1.3), the receiving region has a conductivity  $\sigma_R$ . Let us assume that there are  $M \geq 1$  disjoint receiving antennas so that we can write:

$$\sigma_R = \sum_{m=1}^M \sigma_R^{(m)}. \quad (2.19)$$

Via Ohm's Law  $\mathbf{J}_{\text{out}} = \sigma_R \mathbf{E}$  we determine the current density at the  $m$ -th receiving antenna. Assume that  $S_m$  is the transverse section of the  $m$ -th receiving antenna. Then the current induced in it will be:

$$\begin{aligned} I_{\text{out}}^{(m)} &= \int_{S_m} \sigma_R^{(m)}(\mathbf{x}) \mathbf{E}(\mathbf{x}) \cdot d\mathbf{S} = \sum_{n=1}^N \mathcal{H}_{mn} I_{\text{in}}^{(n)}, \\ \mathcal{H}_{mn} &= \int_{S_m} \sigma_R^{(m)}(\mathbf{x}) \left\{ -j\mu_0\omega(-\Delta + W - k_0^2)^{-1}\mathbf{J}_{\text{in}}^{(n)} \right. \\ &\quad \left. + \nabla \left( \frac{1}{j\epsilon\omega + \sigma_R^{(m)}} \nabla \cdot (-\Delta + W - k_0^2)^{-1}\mathbf{J}_{\text{in}}^{(n)} \right) \right\} \cdot d\mathbf{S}. \end{aligned} \quad (2.20)$$

The transfer matrix elements  $\mathcal{H}_{mn}$  are only frequency dependent and give the current induced in the  $m$ -th receiving antenna when only the  $n$ -th transmitting antenna is fed with current. One must note that the above formula takes into consideration all possible couplings. In the rest of the paper we will try to reduce the complexity of this formula and to arrive to a simpler and practically more convenient expression.

### 3 Decoupling the receivers and transmitters from the environment

It is important to realize that the resolvent  $(-\Delta + W - k_0^2)^{-1}$  contains the whole information about both the electric and magnetic fields. We will express this resolvent and thus  $\mathcal{H}_{mn}$  in a different way, which shows a decoupling between transmitters, receivers and scatterers.

Let us introduce some notation which would describe a situation in which only the transmitter would be present:

$$k_T^2(\mathbf{x}) = k_0^2 + \delta\epsilon_T(\mathbf{x})k_0^2 - j\mu_0\omega\sigma_T(\mathbf{x}) =: k_0^2 + \delta k_T^2(\mathbf{x}), \quad (3.1)$$

where  $\delta k_T^2$  is again smooth and supported only near the transmitter(s). In a similar way we introduce the corresponding quantities for  $M$  and  $R$ .

The perturbation corresponding only to the transmitter(s)  $W_T$  is:

$$W_T \mathbf{A} := -\delta k_T^2 \mathbf{A} + \{\nabla \ln(k_T^2)\} \{\nabla \cdot \mathbf{A}\} \quad (3.2)$$

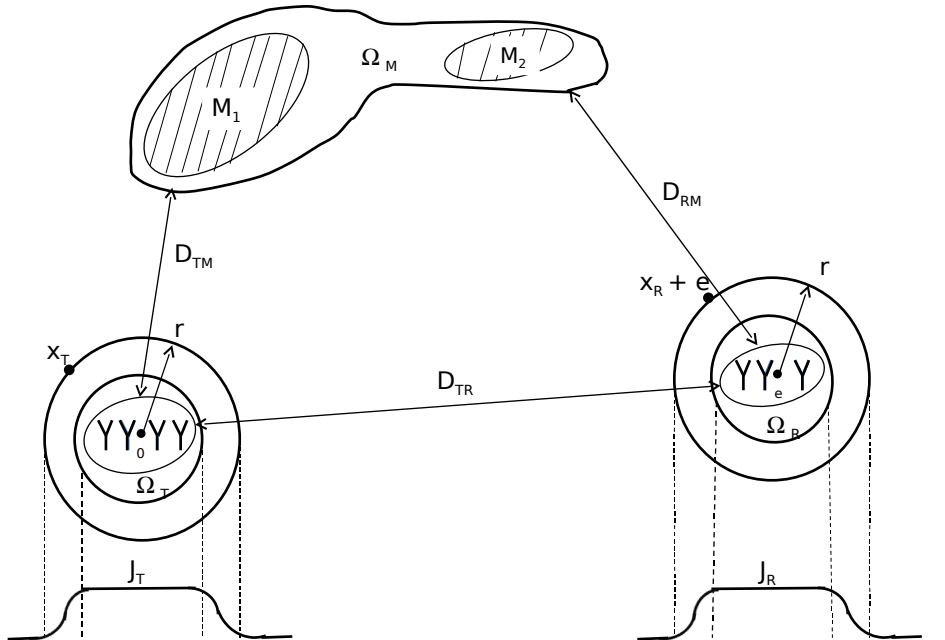
and similar objects can be defined for  $R$  and  $M$ .

Let us introduce the following operators:

$$H = -\Delta + W_T + W_M + W_R, \quad H_T := -\Delta + W_T, \quad H_M := -\Delta + W_M, \quad H_R := -\Delta + W_R. \quad (3.3)$$

Here  $H_T$  only takes into consideration the perturbation induced by the transmitter(s),  $H_M$  does the same thing for the environment, and  $H_R$  for the receiver(s). Assume that both the transmitter(s) and receiver(s) are separated from all other scatterers, and from each other (see figure 1 for what follows).

Figure 1.



If  $\Omega_{T(RM)}$  are bounded open domains completely containing the supports of  $W_{T(RM)}$ , then we quantify this separation by:

$$\begin{aligned} \text{dist}[\Omega_T, \Omega_R] = D_{TR} > 0, \quad \min \{ \text{dist}[\Omega_R, \Omega_M], \text{dist}[\Omega_T, \Omega_M] \} = D_M > 0, \\ r := 1 + \max \{ \text{diam}(\Omega_T), \text{diam}(\Omega_R) \}, \quad r \leq \min \{ D_M/2, D_{TR}/2 \}. \end{aligned} \quad (3.4)$$

Remember that we are interested in the study of the operator  $(H - z)^{-1}$  for  $z = k_0^2 + j0_+$ . Denote by  $\chi_T$ ,  $\chi_R$  and  $\chi_M$  the characteristic functions of  $\Omega_T$ ,  $\Omega_R$  and  $\Omega_M$  respectively. Define a smooth function  $0 \leq J_M \leq 1$  which enters in  $\Omega_T$  and  $\Omega_R$  and "touches" neither the transmitter(s) nor the receiver(s):

$$J_M(\mathbf{x}) = 1, \quad \mathbf{x} \notin \Omega_T \cup \Omega_R, \quad (3.5)$$

$$J_M \chi_{\text{supp}(W_T)} = J_M \chi_{\text{supp}(W_R)} = 0. \quad (3.6)$$

The second condition is possible because the  $\Omega$ 's completely contain the two antenna systems. Let us also note the identity:

$$J_M(\mathbf{x}) \{1 - \chi_T(\mathbf{x}) - \chi_R(\mathbf{x})\} = 1 - \chi_T(\mathbf{x}) - \chi_R(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \quad (3.7)$$

Since we assumed  $r \geq 1$ , we can choose a smooth function  $0 \leq J \leq 1$  such that

$$J(\mathbf{x}) = 1 \quad \text{if} \quad |\mathbf{x}| \leq r - 1, \quad J(\mathbf{x}) = 0 \quad \text{if} \quad |\mathbf{x}| \geq r. \quad (3.8)$$

Take a point in  $\Omega_T$  as the origin of coordinates and let  $\mathbf{e} \in \Omega_R$ . Define

$$J_T(\mathbf{x}) := J(\mathbf{x}), \quad J_R(\mathbf{x}) := J(\mathbf{x} - \mathbf{e}). \quad (3.9)$$

Our choice of  $r$  in (3.4) insures that  $r - 1$  is larger than the diameters of both  $\Omega_T$  and  $\Omega_R$ . Thus:

$$J_T \chi_T = \chi_T, \quad J_R \chi_R = \chi_R. \quad (3.10)$$

The support of any derivative of  $J_T$  is contained in the annulus  $r - 1 \leq |\mathbf{x}| \leq r$ , while the support of any derivative of  $J_R$  is contained in the annulus  $r - 1 \leq |\mathbf{x} - \mathbf{e}| \leq r$ .

Let us denote by  $\chi_{T_a}$  the characteristic function of the spherical annulus  $r - 1 \leq |\mathbf{x}| \leq r$ , and by  $\chi_{R_a}$  the characteristic function of the spherical annulus  $r - 1 \leq |\mathbf{x} - \mathbf{e}| \leq r$ . Then we clearly have the identities:

$$\chi_{T_a}(x) D^\alpha J_T(\mathbf{x}) = D^\alpha J_T(\mathbf{x}), \quad \chi_{R_a}(x) D^\alpha J_R(\mathbf{x}) = D^\alpha J_R(\mathbf{x}). \quad (3.11)$$

The supports of  $\chi_{T_a}$  and  $\chi_{R_a}$  are disjoint from each other, and are situated in the free space. If  $z$  has a sufficiently large imaginary part, we can define the following bounded operator (see (3.3)):

$$S(z) := J_T(H_T - z)^{-1} \chi_T + J_M(H_M - z)^{-1} (1 - \chi_T - \chi_R) + J_R(H_R - z)^{-1} \chi_R. \quad (3.12)$$

We then have:

$$(H - z)S(z) = 1 + K_T(z) + K_M(z) + K_R(z), \quad (3.13)$$

where

$$\begin{aligned} K_T(z) &= [-\Delta, J_T](H_T - z)^{-1} \chi_T, \quad K_M(z) = [-\Delta, J_M](H_M - z)^{-1} (1 - \chi_T - \chi_R), \\ K_R(z) &= [-\Delta, J_R](H_R - z)^{-1} \chi_R, \end{aligned} \quad (3.14)$$

and this is because  $W_{T(RM)}$  commute with  $J_{T(RM)}$ . For example,

$$[W_M, J_M] \mathbf{A} = \{ \nabla \ln(k_M^2) \} \{ \nabla J_M \cdot \mathbf{A} \} = 0$$

due to disjoint support properties of  $\nabla k_M^2$  and  $\nabla J_M$ .  
If  $|\Im(z)|$  is large enough, then one can prove that

$$\max \{ \|K_T\|, \|K_M\|, \|K_R\| \} \leq 1/10,$$

thus

$$\begin{aligned} (H - z)^{-1} &= S(z)[1 + K_T(z) + K_M(z) + K_R(z)]^{-1} \\ &= S(z) - (H - z)^{-1}[K_T(z) + K_M(z) + K_R(z)]. \end{aligned} \quad (3.15)$$

This equality can be extended to all  $z$  where both sides make sense, that is outside of a discrete set of singularities.

Let us observe an important symmetry property coming from time reversal invariance. Define

$$\widetilde{W}\mathbf{A} := -\nabla \left( \{ \nabla \ln(\overline{k^2}) \} \cdot \mathbf{A} \right) - \overline{\delta k^2} \mathbf{A}. \quad (3.16)$$

This operator is depending on frequency through  $k$ , and due to Assumption 1.2 and (2.11) we may write an important identity for the adjoint:

$$\widetilde{W}_{-\omega}^* = W_\omega. \quad (3.17)$$

If  $\widetilde{H}_{-\omega} := -\Delta + \widetilde{W}_{-\omega}$ , then another consequence is:

$$\left\{ (\widetilde{H}_{-\omega} - \bar{z})^{-1} \right\}^* = (H_\omega - z)^{-1}, \quad (3.18)$$

which is also true for  $T, R$  and  $M$  alone.

Write (3.15) with  $\bar{z}$  and  $-\omega$ , then take the adjoint and use (3.18). We obtain:

$$(H_\omega - z)^{-1} = \{S_{-\omega}(\bar{z})\}^* - \{K_T(\bar{z}, -\omega) + K_M(\bar{z}, -\omega) + K_R(\bar{z}, -\omega)\}^* (H_\omega - z)^{-1}. \quad (3.19)$$

We introduce the notations:

$$\tilde{S}(z) := \chi_T(H_T - z)^{-1}J_T + (1 - \chi_T - \chi_R)(H_M - z)^{-1}J_M + \chi_R(H_R - z)^{-1}J_R, \quad (3.20)$$

and

$$\begin{aligned} \tilde{K}_T(z) &:= \{K_T(\bar{z}, -\omega)\}^* = \chi_T(H_T - z)^{-1}[\Delta, J_T], \\ \tilde{K}_M(z) &:= \{K_M(\bar{z}, -\omega)\}^* = (1 - \chi_T - \chi_R)(H_M - z)^{-1}[\Delta, J_M], \\ \tilde{K}_R(z) &:= \{K_R(\bar{z}, -\omega)\}^* = \chi_R(H_R - z)^{-1}[\Delta, J_R]. \end{aligned} \quad (3.21)$$

Then (3.19) can be written in a more compact way:

$$(H - z)^{-1} = \tilde{S}(z) - [\tilde{K}_T(z) + \tilde{K}_M(z) + \tilde{K}_R(z)](H - z)^{-1}. \quad (3.22)$$

Here is the first of our technical results:

**Lemma 3.1.** *The following "almost decoupled" formula holds:*

$$\chi_{\text{supp}(W_R)}(H - z)^{-1}\chi_T = \chi_{\text{supp}(W_R)}\tilde{K}_R(z) \cdot \chi_{R_a}(H - z)^{-1}\chi_{T_a} \cdot K_T(z)\chi_T. \quad (3.23)$$

*Proof.* There are several important things to note here. First, we have the support condition:

$$\text{supp}(\mathbf{J}_{in}) \subset \text{supp}(\chi_T). \quad (3.24)$$

Second, due to the support properties of our various cut-off functions we have  $K_T^2 = K_M^2 = K_R^2 = K_T K_R = K_R K_T = 0$ . Now if we introduce (3.15) in the left hand side of (3.23), we see that the term  $\chi_{\text{supp}(W_R)}S(z)\chi_{\text{supp}(W_T)} = 0$  because of various support properties. We obtain:

$$\chi_{\text{supp}(W_R)}(H - z)^{-1}\chi_T = -\chi_{\text{supp}(W_R)}(H - z)^{-1}K_T(z)\chi_T. \quad (3.25)$$



Now use (3.22) in the right hand side of (3.25). We obtain:

$$\begin{aligned}\chi_{\text{supp}(W_R)}(H - z)^{-1}\chi_T &= -\chi_{\text{supp}(W_R)}\tilde{S}(z)K_T(z)\chi_T \\ &\quad + \chi_{\text{supp}(W_R)}\tilde{K}_R(z)(H - z)^{-1}K_T(z)\chi_T.\end{aligned}\quad (3.26)$$

But the first term on the rhs of the above equality is zero, again due to the supports. Finally, use (3.11) in the above equation, and the proof is over.  $\square$

*Remark.* Note that  $\chi_{\text{supp}(W_R)}(H - z)^{-1}\chi_T$  is the operator entering in (2.20) giving the general transfer matrix element, because  $\chi_T\mathbf{J}_{\text{in}} = \mathbf{J}_{\text{in}}$  and  $\chi_{\text{supp}(W_R)}\sigma_R = \sigma_R$ . Although (3.23) is an **exact** formula, it is not very useful yet because in the middle of the right hand side we still have the full resolvent and not just the resolvent corresponding to the environment. But in the next subsection we will show that  $\chi_{R_a}(H - z)^{-1}\chi_{T_a}$  can be better and better approximated with  $\chi_{R_a}(H_M - z)^{-1}\chi_{T_a}$  if the space occupied by the antennas become smaller and smaller compared to the distances between different objects.

### 3.1 The study of $\chi_{R_a}(H - z)^{-1}\chi_{T_a}$

Remember that  $\chi_{T_a}$  and  $\chi_{R_a}$  are the characteristic functions of two spherical annuli which are at a distance  $r \geq 1$  from both the transmitter(s) and the receiver(s). If the physical space occupied by our antennas becomes very small (mathematically this means that the volume of the supports of  $\chi_T$  and  $\chi_R$  i.e. of  $\Omega_T$  and  $\Omega_R$  is very small), then it would be natural to be able to approximate  $\chi_{R_a}(H - z)^{-1}\chi_{T_a}$  with  $\chi_{R_a}(H_M - z)^{-1}\chi_{T_a}$ . Let us show this here.

**Lemma 3.2.** *Let  $v_r := \text{Vol}(\Omega_R)$  and  $v_t := \text{Vol}(\Omega_T)$ . Then outside a discrete set of frequencies, the vector potential can be approximated in any  $C^k(\Omega_R)$  norm with  $k \geq 0$  in the following way:*

$$\chi_R\mathbf{A} - \tilde{K}_R(k_0^2 + j0_+)\chi_{R_a}[H_M - k_0^2 - j0_+]^{-1}\chi_{T_a}K_T(k_0^2 + j0_+)\mathbf{J}_{\text{in}} = o(\max\{v_t, v_r\}). \quad (3.27)$$

*Proof.* In the Appendix we have formulated a Lippmann-Schwinger type representation of the total resolvent in the presence of  $N$  perturbations, see (5.11). We want to particularize that formula for  $N = 2$  objects. We put  $H_0 = H_M = -\Delta + W_M$ , we take  $W_T$  to be  $W_1$ , and  $W_R$  will be  $W_2$ . The formula (5.11) reads as:

$$R(z) = (H_M - z)^{-1} - \sum_{m=1}^2 \sum_{n=1}^2 (H_M - z)^{-1}\chi_n A_{nm}(z)\chi_m (H_M - z)^{-1}. \quad (3.28)$$

This implies:

$$\begin{aligned}\chi_{R_a}(H - z)^{-1}\chi_{T_a} - \chi_{R_a}(H_M - z)^{-1}\chi_{T_a} \\ = - \sum_{m=1}^2 \sum_{n=1}^2 \chi_{R_a}(H_M - z)^{-1}\chi_n A_{nm}(z)\chi_m (H_M - z)^{-1}\chi_{T_a}.\end{aligned}\quad (3.29)$$

Now the idea is to show that the right hand side of (3.29) is small when the volume of the supports of  $\chi_1$  and  $\chi_2$  are smaller and smaller. We need to estimate the norm of the operator

$$\chi_{R_a}(H_M - z)^{-1}\chi_n A_{nm}(z)\chi_m (H_M - z)^{-1}\chi_{T_a}.$$

The factor  $\chi_{R_a}(H_M - z)^{-1}\chi_n$  is a Hilbert-Schmidt operator and its Hilbert-Schmidt norm tends to zero like  $\sqrt{v_{t,r}}$ . A similar result holds true for  $\chi_m(H_M - z)^{-1}\chi_{T_a}$ . It means that the right hand side of (3.29) is close to zero when the linear dimensions of our antennas become very small.

We can thus write:

$$\chi_{R_a}(H - (k_0^2 + j0_+))^{-1}\chi_{T_a} = \chi_{R_a}(H_M - (k_0^2 + j0_+))^{-1}\chi_{T_a} + \mathcal{O}(\max\{\sqrt{v_t}, \sqrt{v_r}\}), \quad (3.30)$$

outside of a discrete set of frequencies. This proves the lemma for the  $L^2$  norm; one can actually show that this convergence is also true in any  $C^k$  norm; the ingredients are the elliptic regularity and the fact that the supports of  $\chi_T$ ,  $\chi_R$ ,  $\chi_{R_a}$  and  $\chi_{T_a}$  are disjoint. The proof is over.  $\square$

*Remark.* An obvious interpretation of this formula is the following: the input current is transformed into a signal by  $K_T$  (only depending on the transmitter(s)) and sent into the annulus given by  $\chi_{T_a}$ . From there,  $H_M$  scatters the signal into the observation region of the receiver(s), or  $\chi_{R_a}$ . Finally,  $\tilde{K}_R$  takes over the signal and sends it to the receiver(s). Note that the diameters of  $\chi_{T_a}$  and  $\chi_{R_a}$  *do not have to be large*, and this is exactly what happens when some scatterers are close to our antennas. But *the linear dimensions of the emitting and receiving antennas have to be small* in order to be sure that we can approximate  $\chi_{R_a}(H - (k_0^2 + j0_+))^{-1}\chi_{T_a}$  with  $\chi_{R_a}(H_M - (k_0^2 + j0_+))^{-1}\chi_{T_a}$ .

Another important observation: if the linear dimensions of the antennas are important relatively to the other distances in our decomposition, then the expression in (3.27) is not correct. We would need to take into consideration the complete formula for the full resolvent (3.29), because we can no longer ignore the higher order coupling between the emitting and receiving antennas given by the **exact** formula (3.29).

### 3.2 The main theorem

Remember that the transfer matrix in (2.20) is completely characterized by the value of  $\mathbf{A} = (H - k_0^2 - j0_+)^{-1}\mathbf{J}_{\text{in}}$ . In this subsection we assume that the antennas are small enough so that (3.27) makes a good approximation. In order to simplify notation, we write  $z_0$  instead of  $k_0^2 + j0_+$ . All the resolvents we have considered until now are integral operators in the following sense:

$$\{(H - z_0)^{-1}\Psi\}_s(\mathbf{x}) = \sum_{t=1}^3 \int_{\mathbb{R}^3} G_{st}(\mathbf{x}, \mathbf{x}'; z_0) \Psi_t(\mathbf{x}') d\mathbf{x}', \quad s \in \{1, 2, 3\}. \quad (3.31)$$

Their integral kernels are smooth functions of  $\mathbf{x}$  and  $\mathbf{x}'$  outside the diagonal  $\mathbf{x} = \mathbf{x}'$ , due to general elliptic regularity results. We will now express the quantity in (3.27) with the help of the various integral kernels appearing in that equality. Using (3.21) and (3.14) we may write (here  $\mathbf{x} \in \Omega_R$ ) :

$$\begin{aligned} A_{t_1}(\mathbf{x}) \approx & - \sum_{t_2} \int_{\mathbb{R}^3} d\mathbf{u} G_{t_1 t_2}^{(R)}(\mathbf{x}, \mathbf{u}; z_0) \sum_{s_1} \left\{ \frac{\partial}{\partial u_{s_1}} \frac{\partial J_R}{\partial u_{s_1}} + \frac{\partial J_R}{\partial u_{s_1}} \frac{\partial}{\partial u_{s_1}} \right\} \sum_{t_3} \int_{\mathbb{R}^3} d\mathbf{v} G_{t_2 t_3}^{(M)}(\mathbf{u}, \mathbf{v}; z_0) \\ & \cdot \sum_{s_2} \left\{ \frac{\partial}{\partial v_{s_2}} \frac{\partial J_T}{\partial v_{s_2}} + \frac{\partial J_T}{\partial v_{s_2}} \frac{\partial}{\partial v_{s_2}} \right\} \sum_{t_4} \int_{\mathbb{R}^3} d\mathbf{y} G_{t_3 t_4}^{(T)}(\mathbf{v}, \mathbf{y}; z_0) J_{\text{in}, t_4}(\mathbf{y}). \end{aligned} \quad (3.32)$$

In order to simplify the notation, we will assume that the transmitters are located near the origin of coordinates, while the receivers are located near  $\mathbf{e}$ . In this formula we will choose  $J_T$  to be a mollifier of the characteristic function of the ball  $B_r(0) = \{|\mathbf{x}| \leq r\}$ , and  $J_R$  to be a mollifier of the characteristic function of  $B_r(\mathbf{e}) = \{|\mathbf{x} - \mathbf{e}| \leq r\}$ . In  $B_r(0)$  we may choose to work with local spherical coordinates  $(\rho, \hat{\mathbf{x}}_T)$ , with  $\rho \geq 0$  and  $\hat{\mathbf{x}}_T \in S^2$ . The same thing can be done for  $B_r(\mathbf{e})$ , and denote its local spherical coordinates by  $(\rho', \hat{\mathbf{x}}_R)$ . We choose  $J_T$  and  $J_R$  to be radial in these coordinates. Then as they converge towards the characteristic functions, one can prove that the expression in (3.32) converges to:

$$\begin{aligned} A_{t_1}(\mathbf{x}) \approx & -r^4 \sum_{t_2} \int_{S^2} d\hat{\mathbf{x}}_R \int_{\rho'} d\rho' G_{t_1 t_2}^{(R)}(\mathbf{x}, \mathbf{e} + \rho' \hat{\mathbf{x}}_R; z_0) \left\{ \frac{\partial}{\partial \rho'} \delta(\rho' - r) + \delta(\rho' - r) \frac{\partial}{\partial \rho'} \right\} \\ & \sum_{t_3} \int_{S^2} d\hat{\mathbf{x}} \int_{\rho} d\rho G_{t_2 t_3}^{(M)}(\mathbf{e} + \rho' \hat{\mathbf{x}}_R, \rho \hat{\mathbf{x}}_T; z_0) \\ & \left\{ \frac{\partial}{\partial \rho} \delta(\rho - r) + \delta(\rho - r) \frac{\partial}{\partial \rho} \right\} \sum_{t_4} \int_{\mathbb{R}^3} d\mathbf{y} G_{t_3 t_4}^{(T)}(\rho \hat{\mathbf{x}}, \mathbf{y}; z_0) J_{\text{in}, t_4}(\mathbf{y}). \end{aligned} \quad (3.33)$$

Perform the radial integrals and write  $A_{t_1}(\mathbf{x})$  as a sum of four terms:

$$\begin{aligned}
A_{t_1}(\mathbf{x}) \approx & -r^4 \int_{\mathbb{R}^3} d\mathbf{y} \sum_{t_2, t_3, t_4} \int_{S^2} \int_{S^2} d\hat{\mathbf{x}}_T d\hat{\mathbf{x}}_R \\
& \left\{ \left\{ \partial_{\rho'} G_{t_1 t_2}^{(R)}(\mathbf{x}; \mathbf{e} + \rho' \hat{\mathbf{x}}_R; z_0) \right\}_{\rho'=r} \left\{ \partial_{\rho} G_{t_2 t_3}^{(M)}(\mathbf{e} + r \hat{\mathbf{x}}_R; \rho \hat{\mathbf{x}}_T; z_0) \right\}_{\rho=r} G_{t_3 t_4}^{(T)}(r \hat{\mathbf{x}}_T; \mathbf{y}; z_0) \right. \\
& - \left\{ \partial_{\rho'} G_{t_1 t_2}^{(R)}(\mathbf{x}; \mathbf{e} + \rho' \hat{\mathbf{x}}_R; z_0) \right\}_{\rho'=r} G_{t_2 t_3}^{(M)}(\mathbf{e} + r \hat{\mathbf{x}}_R; r \hat{\mathbf{x}}_T; z_0) \left\{ \partial_{\rho} G_{t_3 t_4}^{(T)}(\rho \hat{\mathbf{x}}_T; \mathbf{y}; z_0) \right\}_{\rho=r} \\
& + G_{t_1 t_2}^{(R)}(\mathbf{x}; \mathbf{e} + r \hat{\mathbf{x}}_R; z_0) \left\{ \partial_{\rho'} G_{t_2 t_3}^{(M)}(\mathbf{e} + \rho' \hat{\mathbf{x}}_R; r \hat{\mathbf{x}}_T; z_0) \right\}_{\rho'=r} \left\{ \partial_{\rho} G_{t_3 t_4}^{(T)}(\rho \hat{\mathbf{x}}_T; \mathbf{y}; z_0) \right\}_{\rho=r} \\
& \left. - G_{t_1 t_2}^{(R)}(\mathbf{x}; \mathbf{e} + r \hat{\mathbf{x}}_R; z_0) \left\{ \partial_{\rho', \rho}^2 G_{t_2 t_3}^{(M)}(\mathbf{e} + \rho' \hat{\mathbf{x}}_R, \rho \hat{\mathbf{x}}_T; z_0) \right\}_{\rho=\rho'=r} G_{t_3 t_4}^{(T)}(r \hat{\mathbf{x}}_T; \mathbf{y}; z_0) \right\} \\
& J_{\text{in}, t_4}(\mathbf{y}). \tag{3.34}
\end{aligned}$$

Let us introduce the notation (boldface  $\mathbf{G}$  indicates  $3 \times 3$  matrices):

$$\mathbf{c}_T(\hat{\mathbf{x}}_T; \mathbf{y}) := \begin{pmatrix} \left\{ \partial_{\rho} \mathbf{G}^{(T)}(\rho \hat{\mathbf{x}}_T; \mathbf{y}; z_0) \right\}_{\rho=r} \\ \mathbf{G}^{(T)}(r \hat{\mathbf{x}}_T; \mathbf{y}; z_0) \end{pmatrix}, \quad \mathbf{c}_R(\hat{\mathbf{x}}_R; \mathbf{x}) := \begin{pmatrix} \left\{ \partial_{\rho'} \mathbf{G}^{(R)}(\mathbf{x}, \mathbf{e} + \rho' \hat{\mathbf{x}}_R; z_0) \right\}_{\rho'=r} \\ \mathbf{G}^{(R)}(\mathbf{x}; \mathbf{e} + r \hat{\mathbf{x}}_R; z_0) \end{pmatrix}, \tag{3.35}$$

and

$$\mathcal{M}(\hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T) := \begin{pmatrix} \mathbf{G}^{(M)}(\mathbf{e} + r \hat{\mathbf{x}}_R, r \hat{\mathbf{x}}_T; z_0) & -\left\{ \partial_{\rho} \mathbf{G}^{(M)}(\mathbf{e} + r \hat{\mathbf{x}}_R, \rho \hat{\mathbf{x}}_T; z_0) \right\}_{\rho=r} \\ -\left\{ \partial_{\rho'} \mathbf{G}^{(M)}(\mathbf{e} + \rho' \hat{\mathbf{x}}_R, r \hat{\mathbf{x}}_T; z_0) \right\}_{\rho'=r} & \left\{ \partial_{\rho', \rho}^2 \mathbf{G}^{(M)}(\mathbf{e} + \rho' \hat{\mathbf{x}}_R, \rho \hat{\mathbf{x}}_T; z_0) \right\}_{\rho, \rho'=r} \end{pmatrix}. \tag{3.36}$$

Then the key equation (3.27) can be rewritten as:

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \left\{ (H - z_0)^{-1} \mathbf{J}_{\text{in}} \right\}(\mathbf{x}) \\
&\approx r^4 \int d\mathbf{y} \int_{S^2} \int_{S^2} d\hat{\mathbf{x}}_T d\hat{\mathbf{x}}_R \left\{ \mathbf{c}_R(\hat{\mathbf{x}}_R; \mathbf{x}) \right\}^t \mathcal{M}(\hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T) \mathbf{c}_T(\hat{\mathbf{x}}_T; \mathbf{y}) \mathbf{J}_{\text{in}}(\mathbf{y}), \tag{3.37}
\end{aligned}$$

where the transposition operation is considered with respect to the  $2 \times 2$  structure.

Already here we can see the complete separation between the transmitters, receivers, and the rest of the scatterers in the environment. Introducing this formula back into (2.20), we prove the following theorem:

**Theorem 3.3.** *In the case in which the linear dimensions of the emitting and receiving antennas are small, we can approximate the transfer matrix elements by the formula:*

$$\mathcal{H}_{mn} \approx r^4 \int_{S^2} \int_{S^2} d\hat{\mathbf{x}}_T d\hat{\mathbf{x}}_R \left\langle \mathbf{g}_R^{(m)}(\hat{\mathbf{x}}_R), \mathcal{M}(\hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T) \mathbf{g}_T^{(n)}(\hat{\mathbf{x}}_T) \right\rangle, \tag{3.38}$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual dot-product in  $\mathbb{R}^6$ , while

$$\begin{aligned}
\mathbf{g}_T^{(n)}(\hat{\mathbf{x}}_T) &:= \int \mathbf{c}_T(\hat{\mathbf{x}}_T; \mathbf{y}) \mathbf{J}_{\text{in}}^{(n)}(\mathbf{y}) d\mathbf{y}, \tag{3.39} \\
\mathbf{g}_R^{(m)}(\hat{\mathbf{x}}_R) &:= \int_{S_m} d\mathbf{S}_m \cdot \left\{ -j\mu_0\omega \left\{ \mathbf{c}_R(\hat{\mathbf{x}}_R; \mathbf{x}) \right\}^t + \nabla \left( \frac{1}{j\epsilon\omega + \sigma_R^{(m)}} \nabla \cdot \left\{ \mathbf{c}_R(\hat{\mathbf{x}}_R; \mathbf{x}) \right\}^t \right) \right\} \sigma_R^{(m)}(\mathbf{x})
\end{aligned}$$

are two six-dimensional vectors only depending on the transmitters and receivers respectively. They characterize the radiation pattern of our transmitting and receiving antennas, and they do not change with the environment.

### 3.3 Application: the spread function in the case of distant scatterers

The key mathematical object which completely characterizes how the various scatterers affect the signal is the Green function of the environment  $\mathbf{G}^{(M)}(\mathbf{x}; \mathbf{x}'; k_0^2)$ . We will try to obtain some

relatively simple formulas for this Green function. We have already given in the Appendix a fairly general expression for the resolvent in the presence of  $N$  scatterers, see (5.10) and (5.11).

We would like to give an even simpler, yet generic expression of this Green function when the distance between the scatterers and the two balls  $B_r(0)$  and  $B_r(\mathbf{e})$  is large enough. As before, let us use  $z_0 = k_0^2$ . Assume that the total number of these scatterers is  $S \geq 1$ . The free Green function is given by the well known formula

$$\mathbf{G}_{sp}^{\text{free}}(\mathbf{x}; \mathbf{y}; k_0^2) = \delta_{sp} \frac{e^{jk_0|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}, \quad k_0 = \frac{|\omega|}{c}. \quad (3.40)$$

Let us denote by  $R_0(z_0)$  its associated operator.

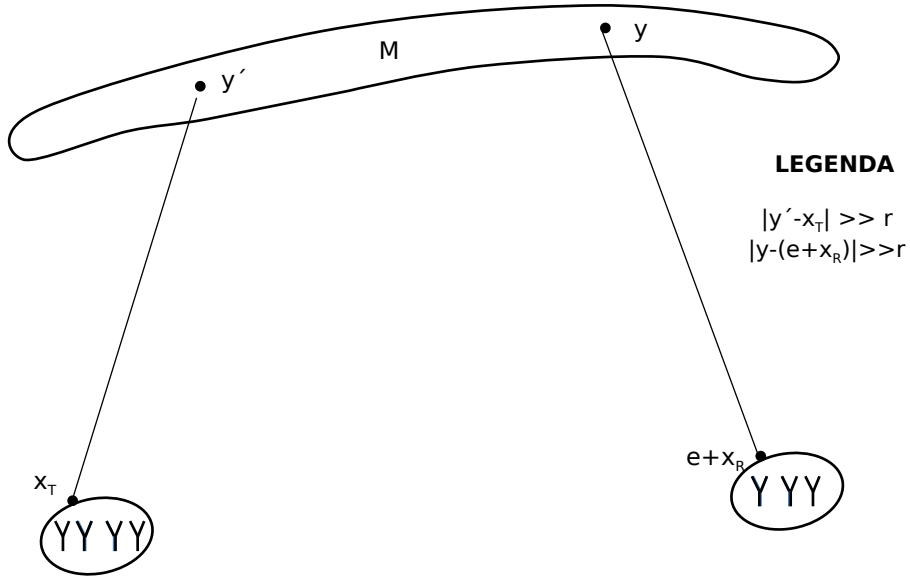
Then (5.11) gives:

$$\begin{aligned} (H_M - z_0)^{-1} &= R_0(z_0) - \sum_{u=1}^S \sum_{v=1}^S R_0(z_0) \chi_u A_{uv}(z_0) \chi_v R_0(z_0) \\ &=: R_0(z_0) - R_0(z_0) T_M(z_0) R_0(z_0), \end{aligned} \quad (3.41)$$

Here the operator  $T_M = \sum_{u=1}^S \sum_{v=1}^S \chi_u A_{uv} \chi_v$  contains all possible interactions between various scatterers present in the environment.

Now assume that the distance between scatterers and the observation points  $r\hat{\mathbf{x}}_T$  and  $\mathbf{e} + r\hat{\mathbf{x}}_R$  is much larger than the radius  $r$  (see figure 2).

Figure 2.



Then if  $\mathbf{y}$  and  $\mathbf{y}'$  are in the support of the scatterers we may write:

$$|r\hat{\mathbf{x}}_T - \mathbf{y}'| = |\mathbf{y}'| - r\hat{\mathbf{x}}_T \cdot \hat{\mathbf{y}}' + \mathcal{O}(|\mathbf{y}'|^{-1}), \quad \hat{\mathbf{y}}' = \frac{\mathbf{y}'}{|\mathbf{y}'|}$$

and

$$|\mathbf{e} + r\hat{\mathbf{x}}_R - \mathbf{y}| = |\mathbf{y} - \mathbf{e}| - r\hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y} - \mathbf{e})} + \mathcal{O}(|\mathbf{y} - \mathbf{e}|^{-1}).$$

Therefore

$$\begin{aligned}\mathbf{G}_{sp}^{\text{free}}(\mathbf{e} + r\hat{\mathbf{x}}_R; \mathbf{y}; k_0^2) &= \delta_{sp} \frac{e^{jk_0|\mathbf{y}-\mathbf{e}|} e^{-jk_0 r\hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y}-\mathbf{e})}}}{4\pi|\mathbf{y}-\mathbf{e}|} + \mathcal{O}(|\mathbf{y}-\mathbf{e}|^{-2}), \\ \mathbf{G}_{sp}^{\text{free}}(\mathbf{y}'; r\hat{\mathbf{x}}_T; k_0^2) &= \delta_{sp} \frac{e^{jk_0|\mathbf{y}'|} e^{-jk_0 r\hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}'}}}{4\pi|\mathbf{y}'|} + \mathcal{O}(|\mathbf{y}'|^{-2}).\end{aligned}\quad (3.42)$$

Using some elliptic regularity estimates and integration by parts, one can prove the existence of some  $3 \times 3$  matrices  $\mathbf{a}_{uv}(\mathbf{y}_u, \mathbf{y}_v; z_0)$  and  $\mathbf{b}_u(\mathbf{y}_u; z_0)$  which are jointly continuous as functions of  $\mathbf{y}_u$  and  $\mathbf{y}_v$  on the support of  $\chi_u$  and  $\chi_v$  such that:

$$\begin{aligned}& \{R_0(z_0)\chi_u A_{uv}(z_0)\chi_v R_0(z_0)\}(\mathbf{e} + r\hat{\mathbf{x}}_R, r\hat{\mathbf{x}}_T) \\ &= \int_{\text{supp}(\chi_u)} \int_{\text{supp}(\chi_v)} \frac{e^{-jk_0 r\hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y}_u - \mathbf{e})}} e^{-jk_0 r\hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}_v}}}{16\pi^2|\mathbf{y}_u - \mathbf{e}| |\mathbf{y}_v|} \{\mathbf{a}_{uv}(\mathbf{y}_u, \mathbf{y}_v; z_0) + \mathbf{b}_u(\mathbf{y}_u; z_0)\delta(\mathbf{y}_u - \mathbf{y}_v)\} d\mathbf{y}_u d\mathbf{y}_v \\ &+ \mathcal{O}(1/D_M^3),\end{aligned}\quad (3.43)$$

where  $D_M$  is as in (3.4) (i.e. the minimal distance between the emitters/receivers and the scatterers of the environment). The matrices  $\mathbf{a}_{uv}(\cdot, \cdot; z_0)$  and  $\mathbf{b}_u(\cdot; z_0)$  contain the full scattering information and depend on all scatterers not just on  $u$  and  $v$ . Note though that the dependence of  $\mathbf{x}$  and  $\mathbf{x}'$  is now very explicit.

Using (3.43) in (3.41) allows us to introduce a single  $3 \times 3$  matrix kernel  $\mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0)$  which contains the full scattering information of the medium:

$$\begin{aligned}\mathbf{G}^{(M)}(\mathbf{e} + r\hat{\mathbf{x}}_R, r\hat{\mathbf{x}}_T; k_0^2) &= \mathbf{G}^{\text{free}}(\mathbf{e} + r\hat{\mathbf{x}}_R, r\hat{\mathbf{x}}_T; k_0^2) \\ &- \int_{\text{supp}(\chi_M)} \int_{\text{supp}(\chi_M)} \frac{e^{-jr k_0 \hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y}-\mathbf{e})}} e^{-jr k_0 \hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}'}}}{16\pi^2|\mathbf{y}-\mathbf{e}| |\mathbf{y}'|} \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) d\mathbf{y} d\mathbf{y}' + \mathcal{O}(1/D_M^3),\end{aligned}\quad (3.44)$$

where  $\chi_M$  is the characteristic function of all scatterers, while  $\mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0)$  has the structure:

$$\mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) = \tau(\mathbf{y}, \mathbf{y}'; z_0) + \theta(\mathbf{y}; z_0)\delta(\mathbf{y} - \mathbf{y}')$$

where  $\tau(\cdot, \cdot; z_0)$  and  $\theta(\cdot; z_0)$  are continuous on their domain of definition. These functions play the role of a scattering kernel, and contain the full scattering information of the medium.

When we introduce this formula in (3.36) we obtain two contributions: one from the "empty" space and another one coming from the scatterers. The same happens with the transfer matrix.

This scattering generated contribution can be expressed as:

$$\mathcal{H}_{mn}^{\text{scatt}} \approx r^4 \int_{S^2} \int_{S^2} d\hat{\mathbf{x}}_T d\hat{\mathbf{x}}_R \int_{\text{supp}(\chi_M)} \int_{\text{supp}(\chi_M)} d\mathbf{y} d\mathbf{y}' \left\langle \mathbf{g}_R^{(m)}(\hat{\mathbf{x}}_R), \mathcal{M}(\mathbf{y}, \mathbf{y}'; \hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T) \mathbf{g}_T^{(n)}(\hat{\mathbf{x}}_T) \right\rangle \quad (3.45)$$

where  $\mathcal{M}(\mathbf{y}, \mathbf{y}'; \hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T)$  is constructed by introducing the scattering contribution from (3.44) in (3.36) which gives:

$$\begin{aligned}\mathcal{M}(\mathbf{y}, \mathbf{y}'; \hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T) &:= - \frac{e^{-jr k_0 \hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y}-\mathbf{e})}} e^{-jr k_0 \hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}'}}}{16\pi^2|\mathbf{y}-\mathbf{e}| |\mathbf{y}'|} \\ &\times \begin{pmatrix} \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) & j k_0 (\hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}'}) \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) \\ j k_0 \hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y}-\mathbf{e})} \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) & -k_0^2 (\hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}'}) (k_0 \hat{\mathbf{x}}_R \cdot \widehat{(\mathbf{y}-\mathbf{e})}) \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) \end{pmatrix}.\end{aligned}\quad (3.46)$$

We are now ready to formulate the main result of this section:

**Corollary 3.4.** *The scattering contribution  $\mathcal{H}_{mn}^{\text{scatt}}$  to the transfer matrix element can be expressed as*

$$\mathcal{H}_{mn}^{\text{scatt}} = \int \int_{S^2 \times S^2} d\Omega_R d\Omega_T \left\langle \mathbf{h}_R^{(m)}(\Omega_R), \mathcal{A}(\Omega_R, \Omega_T) \mathbf{h}_T^{(n)}(\Omega_T) \right\rangle + \mathcal{O}(1/D_M^3), \quad (3.47)$$

where  $\mathbf{h}_T^{(n)}(\Omega_T)$  is a six dimensional vector which can be interpreted as the signal sent by the transmitter in the direction  $\Omega_T$ , then  $\mathcal{A}(\Omega_R, \Omega_T)$  is a  $6 \times 6$  spread matrix only depending on the scatterers, and finally  $\mathbf{h}_R^{(m)}(\Omega_R)$  is a six dimensional vector describing what the receiver got from the direction  $\Omega_R$ .

*Proof.* Going back to (3.45) we may assume that:

$$\mathbf{g}_T^{(n)}(\hat{\mathbf{x}}_T) = \begin{pmatrix} \mathbf{F}_1(\hat{\mathbf{x}}_T) \\ \mathbf{F}_2(\hat{\mathbf{x}}_T) \end{pmatrix}, \quad \mathbf{g}_R^{(m)}(\hat{\mathbf{x}}_R) = \begin{pmatrix} \mathbf{H}_1(\hat{\mathbf{x}}_R) & \mathbf{H}_2(\hat{\mathbf{x}}_R) \end{pmatrix} \quad (3.48)$$

where the  $\mathbf{F}$ 's and  $\mathbf{H}$ 's are some three dimensional vectors. Then we have:

$$\begin{aligned} \left\langle \mathbf{g}_R^{(m)}(\hat{\mathbf{x}}_R), \mathcal{M}(\mathbf{y}, \mathbf{y}'; \hat{\mathbf{x}}_R; \hat{\mathbf{x}}_T) \mathbf{g}_T^{(n)}(\hat{\mathbf{x}}_T) \right\rangle &= - \frac{e^{-jrk_0 \hat{\mathbf{x}}_R \cdot (\widehat{\mathbf{y} - \mathbf{e}})} - e^{-jrk_0 \hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}}'}}{16\pi^2 |\mathbf{y} - \mathbf{e}| |\mathbf{y}'|} \\ &\times \{ \mathbf{H}_1(\hat{\mathbf{x}}_R) \cdot \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) \mathbf{F}_1(\hat{\mathbf{x}}_T) + \mathbf{H}_1(\hat{\mathbf{x}}_R) \cdot \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) \mathbf{F}_2(\hat{\mathbf{x}}_T) jk_0(\hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}}') \\ &+ jk_0(\hat{\mathbf{x}}_R \cdot \widehat{\mathbf{y} - \mathbf{e}}) \mathbf{H}_2(\hat{\mathbf{x}}_R) \cdot \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) \mathbf{F}_1(\hat{\mathbf{x}}_T) \\ &- k_0^2(\hat{\mathbf{x}}_T \cdot \widehat{\mathbf{y}}')(\hat{\mathbf{x}}_R \cdot \widehat{\mathbf{y} - \mathbf{e}}) \mathbf{H}_2(\hat{\mathbf{x}}_R) \cdot \mathbf{t}_M(\mathbf{y}, \mathbf{y}'; z_0) \mathbf{F}_2(\hat{\mathbf{x}}_T) \}. \end{aligned} \quad (3.49)$$

Now we integrate with respect to the angles  $\hat{\mathbf{x}}_R$  and  $\hat{\mathbf{x}}_T$  in (3.45). It is useful to define local spherical coordinates near both the transmitters and the receivers. Introduce  $\Omega_T := \widehat{\mathbf{y}'}$  and  $s_T := |\mathbf{y}'|$ , and  $\Omega_R := \widehat{\mathbf{y} - \mathbf{e}}$  and  $s_R := |\mathbf{y} - \mathbf{e}|$ . Now we can define:

$$\mathbf{h}_T^{(n)}(\Omega_T) := \begin{pmatrix} \int_{S^2} e^{-jrk_0 \hat{\mathbf{x}}_T \cdot \Omega_T} \mathbf{F}_1(\hat{\mathbf{x}}_T) d\hat{\mathbf{x}}_T \\ \int_{S^2} e^{-jrk_0 \hat{\mathbf{x}}_T \cdot \Omega_T} jk_0(\hat{\mathbf{x}}_T \cdot \Omega_T) \mathbf{F}_2(\hat{\mathbf{x}}_T) d\hat{\mathbf{x}}_T \end{pmatrix}, \quad (3.50)$$

$$\mathbf{h}_R^{(m)}(\Omega_R) := \begin{pmatrix} \int_{S^2} e^{-jrk_0 \hat{\mathbf{x}}_R \cdot \Omega_R} \mathbf{H}_1(\hat{\mathbf{x}}_R) d\hat{\mathbf{x}}_R, & \int_{S^2} e^{-jrk_0 \hat{\mathbf{x}}_R \cdot \Omega_R} jk_0(\hat{\mathbf{x}}_R \cdot \Omega_R) \mathbf{H}_2(\hat{\mathbf{x}}_R) d\hat{\mathbf{x}}_R \end{pmatrix}, \quad (3.51)$$

and

$$\mathcal{A}(\Omega_R, \Omega_T) := - \int_0^\infty ds_R \int_0^\infty ds_T \frac{r^4 s_R s_T}{16\pi^2} \begin{pmatrix} \mathbf{t}_M(s_R, \Omega_R; s_T, \Omega_T; z_0) & \mathbf{t}_M(s_R, \Omega_R; s_T, \Omega_T; z_0) \\ \mathbf{t}_M(s_R, \Omega_R; s_T, \Omega_T; z_0) & \mathbf{t}_M(s_R, \Omega_R; s_T, \Omega_T; z_0) \end{pmatrix}. \quad (3.52)$$

Then (3.47) follows immediately. Note that the integrand in the double integral defining the spread matrix  $\mathcal{A}(\Omega_R, \Omega_T)$  is different from zero only on the compact radial supports of the total scatterer defined by the intersection of the scatterer with the direction  $\Omega_T$  seen by an observer placed at the origin and by the intersection of the scatterer with the direction  $\Omega_R$  seen by an observer placed at  $\mathbf{e}$ . □

## 4 Conclusions and open problems

1. Starting from the Maxwell equations and the Ohm's Law we give a rigorous analysis of the input-output relationship of a MIMO system as a direct, well-posed problem. The main observation is that we can decouple the group of transmitters from the group of receivers and scatterers if the linear dimensions of all our antennas are small enough, see (3.27) and

the discussion around it. Even if this smallness condition would not be satisfied, we could in theory give higher order corrections with respect to the decoupled case.

We stress that the transmitting (receiving) antennas are allowed to interfere among themselves, so in principle the coupling in between the transmitting (receiving) antennas is taken into account.

2. In the decoupled case, we can analyze the transfer matrix and identify in it the spread kernel due to the environment alone. The most important result of our paper is contained in the equations (3.38) and (3.39); there we do not need the scatterers to be located in the far-field region of the receiving and transmitting antennas. In the particular case in which the scatterers are far away, then our formulas simplify and we can recover previously known empirical results derived in [8, 19, 6, 7, 12]; in this case, our results are given in (3.50)-(3.52).

In a future work we will investigate the behavior of the spread kernel as a function of angles and frequency for scatterers which are not necessarily far away from the emitting and receiving antennas.

3. Our formalism does not (yet) allow ideal metals ( $\sigma = \infty$ ) or discontinuities in  $\epsilon$ 's. Investigating how our formalism behaves when one takes the limit of non-smooth coefficients is a very interesting and difficult problem of operator theory and functional analysis, which will be investigated elsewhere.
4. We have not elaborated on the question of the limiting absorption principle formulated in (2.16); complete proofs based on the analytic Fredholm alternative will be given elsewhere.
5. We think that our new understanding of the construction of the transfer matrix  $\mathcal{H}$  paves the way for the study of the behavior of the capacity in the case the number of antennas grows in a definite volume (see [19] for a discussion on this subject).

## 5 Appendix: A Lippmann-Schwinger type equation for the resolvent

Let us consider an operator  $H = H_0 + \sum_{k=1}^N W_k$  where  $H_0$  is a "nice" elliptic second order reference differential operator (the typical example is  $-\Delta$ ), and the perturbations  $W_k$  are first order differential operator with smooth coefficients. The supports of the coefficients of  $W_k$  are disjoint from those of the coefficients of  $W_j$  if  $k \neq j$ . Denote by  $\chi_m$  the characteristic function of a ball completely containing the support of the coefficients of  $W_m$ . Denote by  $D_{mn}$  the distance between the supports of  $\chi_m$  and  $\chi_n$ .

Denote by  $R(z) := (H - z)^{-1}$  and by  $R_0(z) := (H_0 - z)^{-1}$  whenever the two inverses (resolvents) exist.

### 5.1 The case of just one scatterer

Assume  $N = 1$ . Choose  $z$  in the resolvent set of  $H_0$ . If the imaginary part of  $z$  is large enough, then  $\|W_1(H_0 - z)^{-1}\| < 1$  and  $(H - z)^{-1}$  exists as a bounded operator in  $(L^2(\mathbb{R}^3))^3$ . The second resolvent identity reads as:

$$R(z) = R_0(z) - R_0(z)W_1R(z). \quad (5.1)$$

Multiply the above equation with  $W_1$  at the left, and write

$$W_1R = W_1R_0 - W_1R_0W_1R.$$

Using that  $\chi_1W_1 = W_1$  we have:

$$W_1R = (\text{Id}_1 + W_1R_0\chi_1)^{-1}W_1R_0, \quad (5.2)$$

where  $\text{Id}_1$  is the identity operator in  $(L^2(\text{supp}(\chi_1)))^3$ . The inverse  $(\text{Id}_1 + W_1 R_0 \chi_1)^{-1}$  (if it exists) is to be taken in  $(L^2(\text{supp}(\chi_1)))^3$ . We will always assume the existence of this inverse; generically this is true if  $W_1$  is relatively compact to  $R_0$  and one can apply the Fredholm alternative.

Note that if we know  $W_1 R$ , then we know  $R$  everywhere in the space because we can replace (5.2) in (5.1) and obtain:

$$\begin{aligned} R(z) &= R_0(z) - R_0(z) \chi_1 (\text{Id}_1 + W_1 R_0(z) \chi_1)^{-1} W_1 R_0(z) \\ &= R_0(z) - R_0(z) T_1(z) R_0(z), \\ T_1(z) &:= \chi_1 (\text{Id}_1 + W_1 R_0(z) \chi_1)^{-1} W_1. \end{aligned} \quad (5.3)$$

## 5.2 The case of several scatterers

Let  $N \geq 2$ . The equivalent of (5.1) reads as:

$$R(z) = R_0(z) - \sum_{m=1}^N R_0(z) W_m R(z). \quad (5.4)$$

We multiply with  $W_n$  at the left on both sides of the above equality and obtain:

$$W_n R(z) = W_n R_0(z) - \sum_{m=1}^N W_n R_0(z) W_m R(z) = \sum_{m=1}^N \{\delta_{nm} - W_n R_0(z) \chi_m\} W_m R(z). \quad (5.5)$$

Denote by  $\mathcal{H}_N := \oplus_{k=1}^N (L^2(\text{supp}(\chi_k)))^3$ . Define the bounded operator  $\mathcal{M}$  given by

$$\begin{aligned} \mathcal{H}_N \ni \Psi = \oplus_{k=1}^N \psi_k &\rightarrow \mathcal{M}(z) \Psi = \oplus_{i=1}^N \left\{ \sum_{k=1}^N \mathcal{M}_{ik} \psi_k \right\}, \\ \mathcal{M}_{ik}(z) &:= W_i R_0(z) \chi_k. \end{aligned} \quad (5.6)$$

Define the immersion operator

$$\begin{aligned} J_N : (L^2(\mathbb{R}^3))^3 &\mapsto \mathcal{H}_N, \quad (L^2(\mathbb{R}^3))^3 \ni \phi \rightarrow J\psi = \oplus_{n=1}^N \chi_n \phi, \\ J_N^* : \mathcal{H}_N &\mapsto (L^2(\mathbb{R}^3))^3, \quad \mathcal{H}_N \ni \Psi = \oplus_{n=1}^N \psi_n \mapsto J_N^* \Psi = \sum_{n=1}^N \chi_n \psi_n. \end{aligned} \quad (5.7)$$

Denoting the total perturbation by  $W = \sum_{n=1}^N W_n$ , the equation (5.5) can be seen as:

$$J_N W R(z) = J_N W R_0(z) - \mathcal{M}(z) J_N W R(z). \quad (5.8)$$

Denote by  $\text{Id}$  the identity operator in  $\mathcal{H}_N$ . If  $\text{Id} + \mathcal{M}(z)$  is invertible in  $\mathcal{H}_N$ , then the above equation can be rewritten as:

$$J_N W R(z) = \{\text{Id} + \mathcal{M}(z)\}^{-1} J_N W R_0(z). \quad (5.9)$$

Moreover, equation (5.4) can be written as:

$$\begin{aligned} R(z) &= R_0(z) - R_0(z) J_N^* J_N W R(z) \\ &= R_0(z) - R_0(z) J_N^* \{\text{Id} + \mathcal{M}(z)\}^{-1} J_N W R_0(z) \end{aligned} \quad (5.10)$$

where in the second line we used (5.9). It follows that the full resolvent can be always written as:

$$R(z) = R_0(z) - \sum_{n=1}^N \sum_{m=1}^N R_0(z) \chi_n A_{nm}(z) \chi_m R_0(z), \quad (5.11)$$



where the operators  $A_{nm}(z)$  act in  $(L^2(\mathbb{R}^3))^3$ .

Now let us assume that the distance  $D_{mn}$  between any two different supports is large. We can split the operator  $\mathcal{M}$  in a diagonal and off-diagonal part:

$$\mathcal{M}_d(z) := \oplus_{n=1}^N W_n R_0(z) \chi_n, \quad \mathcal{M}_o(z) := \mathcal{M}(z) - \mathcal{M}_d(z). \quad (5.12)$$

Clearly, the norm of  $\mathcal{M}_o(z)$  becomes smaller and smaller when  $D_{mn}$  becomes larger. For example, if  $H_0 = -\Delta$  we have that  $\|W_m R_0(z) \chi_n\| \leq C/D_{mn}$ . Thus if the minimal distance between any two scatterers becomes larger than a critical value, we can write:

$$\{\text{Id} + \mathcal{M}(z)\}^{-1} = \sum_{i=0}^{\infty} (-1)^i \{\text{Id} + \mathcal{M}_d(z)\}^{-1} \{\mathcal{M}_o(z) [\text{Id} + \mathcal{M}_d(z)]^{-1}\}^i. \quad (5.13)$$

The operators  $[\text{Id} + \mathcal{M}_d(z)]^{-1}$  are diagonal and given by:

$$[\text{Id} + \mathcal{M}_d(z)]^{-1} = \oplus_{n=1}^N [\text{Id}_n + W_n R_0(z) \chi_n]^{-1}. \quad (5.14)$$

Making the analogy with (5.3) we introduce the notation

$$T_n(z) := [\text{Id}_n + W_n R_0(z) \chi_n]^{-1} W_n, \quad (5.15)$$

where  $T_n$  is a bounded operator in  $(L^2(\text{supp}(\chi_n)))^3$ . Then introducing (5.15) and (5.13) in (5.10) we obtain (we drop the  $z$  dependence for simplicity):

$$R = R_0 - \sum_{n=1}^N R_0 T_n R_0 - \sum_{i=1}^{\infty} (-1)^i \sum_{m_0, m_1, \dots, m_i=1}^N R_0 T_{m_0} R_0 T_{m_1} \cdots R_0 T_{m_i} R_0, \quad (5.16)$$

where the symbol  $\sum$  means that the sum is performed on indices which obey  $m_0 \neq m_1, m_1 \neq m_2, \dots, m_{i-1} \neq m_i$ .

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